

# Effects of a mixed vector-scalar screened Coulomb potential for spinless particles

Antonio S. de Castro

UNESP - Campus de Guaratinguetá  
Departamento de Física e Química  
Caixa Postal 205  
12516-410 Guaratinguetá SP - Brasil

E-mail address: castro@feg.unesp.br (A.S. de Castro)

## Abstract

The problem of a spinless particle subject to a general mixing of vector and scalar screened Coulomb potentials in a two-dimensional world is analyzed and its bounded solutions are found. Some unusual results, including the existence of a bona fide solitary zero-eigenmode solution, are revealed for the Klein-Gordon equation. The cases of pure vector and scalar potentials, already analyzed in previous works, are obtained as particular cases.

# 1 Introduction

In a two-dimensional space-time the screened Coulomb potential ( $\sim e^{-|x|/\lambda}$ ) has been analyzed and its analytical solutions have been found for the Dirac equation with vector [1], scalar [2] and pseudoscalar [3] couplings and for the Klein-Gordon (KG) equation with vector [4] and scalar [5] couplings. As has been emphasized in Refs. [4] and [5], the solution of the KG equation with this sort of potential may find applications in the study of pionic atoms, doped Mott insulators, doped semiconductors, interaction between ions, quantum dots surrounded by a dielectric or a conducting medium, protein structures, etc.

In the present work the problem of a spinless particle in the background of a screened Coulomb potential is considered with a general mixing of vector and scalar Lorentz structures. This sort of mixing beyond its potential physical applications, shows to be a powerful tool to obtain a deeper insight about the nature of the KG equation and its solutions. The problem is mapped into an exactly solvable Sturm-Liouville problem of a Schrödinger-like equation with an effective symmetric Morse-like potential, or an effective screened Coulomb potential in particular circumstances. The cases of pure vector and scalar potentials, already analyzed in [4]-[5], are obtained as particular cases.

In the presence of vector and scalar potentials the 1+1 dimensional time-independent KG equation for a spinless particle of rest mass  $m$  reads

$$-\hbar^2 c^2 \frac{d^2\psi}{dx^2} + (mc^2 + V_s)^2 \psi = (E - V_v)^2 \psi \quad (1)$$

where  $E$  is the energy of the particle,  $c$  is the velocity of light and  $\hbar$  is the Planck constant. The vector and scalar potentials are given by  $V_v$  and  $V_s$ , respectively. The subscripts for the terms of potential denote their properties under a Lorentz transformation:  $v$  for the time component of the 2-vector potential and  $s$  for the scalar term. It is worth to note that the KG equation is covariant under  $x \rightarrow -x$  if  $V_v(x)$  and  $V_s(x)$  remain the same. Also note that  $\psi$  remains invariant under the simultaneous transformations  $E \rightarrow -E$  and  $V_v \rightarrow -V_v$ . Furthermore, for  $V_v = 0$ , the case of a pure scalar potential, the negative- and positive-energy levels are disposed symmetrically about  $E = 0$ .

The KG equation can also be written as

$$H_{eff}\psi = -\frac{\hbar^2}{2m} \psi'' + V_{eff} \psi = E_{eff} \psi \quad (2)$$

where

$$E_{eff} = \frac{E^2 - m^2 c^4}{2mc^2}, \quad V_{eff} = \frac{V_s^2 - V_v^2}{2mc^2} + V_s + \frac{E}{mc^2} V_v \quad (3)$$

From this one can see that for potentials which tend to  $\pm\infty$  as  $|x| \rightarrow \infty$  it follows that  $V_{eff} \rightarrow (V_s^2 - V_v^2) / (2mc^2)$ , so that the KG equation furnishes a purely discrete (continuum) spectrum for  $|V_s| > |V_v|$  ( $|V_s| < |V_v|$ ). On the other hand, if the potentials vanish as  $|x| \rightarrow \infty$  the continuum spectrum is omnipresent but the necessary conditions for the existence of a discrete spectrum is not an easy task for general functional forms. The boundary conditions on the eigenfunctions come into existence by demanding that the effective Hamiltonian given (2) is Hermitian, viz.

$$\int_a^b dx \psi_n^* (H_{eff} \psi_{n'}) = \int_a^b dx (H_{eff} \psi_n)^* \psi_{n'} \quad (4)$$

where  $\psi_n$  is an eigenfunction corresponding to an effective eigenvalue  $(E_{eff})_n$  and  $(a, b)$  is the interval under consideration. In passing, note that a necessary consequence of Eq. (4) is that the eigenfunctions corresponding to distinct effective eigenvalues are orthogonal. It can be shown that (4) is equivalent to

$$\left[ \psi_n^* \frac{d\psi_{n'}}{dx} - \frac{d\psi_n^*}{dx} \psi_{n'} \right]_{x=a}^{x=b} = 0 \quad (5)$$

In the nonrelativistic approximation (potential energies small compared to  $mc^2$  and  $E \simeq mc^2$ ) Eq. (1) becomes

$$\left( -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V_v + V_s \right) \psi = (E - mc^2) \psi \quad (6)$$

so that  $\psi$  obeys the Schrödinger equation with binding energy equal to  $E - mc^2$  without distinguishing the contributions of vector and scalar potentials.

It is remarkable that the KG equation with a scalar potential, or a vector potential contaminated with some scalar coupling, is not invariant under  $V \rightarrow V + const.$ , this is so because only the vector potential couples to the positive-energies in the same way it couples to the negative-ones, whereas the scalar potential couples to the mass of the particle. Therefore, if there is any scalar coupling the absolute values of the energy will have physical significance and the freedom to choose a zero-energy will be lost. It is well known that a confining potential in the nonrelativistic approach is not confining in the relativistic approach when it is considered as a Lorentz vector. It is surprising

that relativistic confining potentials may result in nonconfinement in the non-relativistic approach. This last phenomenon is a consequence of the fact that vector and scalar potentials couple differently in the KG equation whereas there is no such distinction among them in the Schrödinger equation. This observation permit us to conclude that even a “repulsive” potential can be a confining potential. The case  $V_v = -V_s$  presents bounded solutions in the relativistic approach, although it reduces to the free-particle problem in the nonrelativistic limit. The attractive vector potential for a particle is, of course, repulsive for its corresponding antiparticle, and vice versa. However, the attractive (repulsive) scalar potential for particles is also attractive (repulsive) for antiparticles. For  $V_v = V_s$  and an attractive vector potential for particles, the scalar potential is counterbalanced by the vector potential for antiparticles as long as the scalar potential is attractive and the vector potential is repulsive. As a consequence there is no bounded solution for antiparticles. For  $V_v = 0$  and a pure scalar attractive potential, one finds energy levels for particles and antiparticles arranged symmetrically about  $E = 0$ . For  $V_v = -V_s$  and a repulsive vector potential for particles, the scalar and the vector potentials are attractive for antiparticles but their effects are counterbalanced for particles. Thus, recurring to this simple standpoint one can anticipate in the mind that there is no bound-state solution for particles in this last case of mixing.

## 2 The mixed vector-scalar screened Coulomb potential

Now let us focus our attention on scalar and vector potentials in the form

$$V_s = -\frac{g_s}{2\lambda} \exp\left(-\frac{|x|}{\lambda}\right), \quad V_v = -\frac{g_v}{2\lambda} \exp\left(-\frac{|x|}{\lambda}\right) \quad (7)$$

where the coupling constants,  $g_s$  and  $g_v$ , are dimensionless real parameters and  $\lambda$ , related to the range of the interaction, is a positive parameter. In this case the second equation of (3) transmutes into

$$V_{eff} = V_1 \exp\left(-\frac{|x|}{\lambda}\right) + V_2 \exp\left(-2\frac{|x|}{\lambda}\right) \quad (8)$$

where

$$V_1 = -\frac{1}{2\lambda} \left( g_s + \frac{E}{mc^2} g_v \right), \quad V_2 = \frac{g_s^2 - g_v^2}{8\lambda^2 mc^2} \quad (9)$$

Therefore, one has to search for bounded solutions in an effective symmetric Morse-like potential for  $g_s^2 \neq g_v^2$ , or screened Coulomb potential for  $g_s^2 = g_v^2$ . The KG eigenvalues are obtained by inserting the effective eigenvalues into the first equation of (3). Since the effective potential is even under  $x \rightarrow -x$ , the KG eigenfunction can be expressed as a function of definite parity. Thus, we can concentrate our attention on the positive half-line and impose boundary conditions on  $\psi$  at  $x = 0$  and  $x = +\infty$ . From (5) one can see that in addition to  $\psi(\infty) = 0$ , the boundary conditions can be met in two distinct ways: the odd function obeys the Neumann condition at the origin ( $d\psi/dx|_{x=0} = 0$ ) whereas the even function obeys the Dirichlet condition ( $\psi(0) = 0$ ).

Note carefully that the potentials  $V_s$  and  $V_v$  vanish as  $|x| \rightarrow \infty$  and the KG equation can furnish a discrete spectrum when  $V_1 < 0$  and  $V_2 \geq 0$ , or  $V_1 < |V_2|$  and  $V_2 < 0$ . Only in those circumstances the effective potentials present potential-well structures permitting bounded solutions in the range  $|E| < mc^2$ . The eigenenergies in the range  $|E| > mc^2$  correspond to the continuum.

Now we move to consider a quantitative treatment of our problem by considering the two distinct classes of effective potentials.

## 2.1 The effective screened Coulomb potential ( $g_s^2 = g_v^2$ )

For this class of effective potential, the discrete spectrum arises when  $V_1 < 0$  and  $V_2 = 0$ , corresponding to  $g_s [1 + \text{sgn}(g_v) E / (mc^2)] > 0$  and  $g_s = |g_v|$ . Defining the dimensionless quantities

$$\begin{aligned} y &= y_0 \exp\left(-\frac{|x|}{2\lambda}\right), & y_0 &= \frac{2}{\hbar} \sqrt{\lambda m g_s \left[1 + \frac{E}{mc^2} \text{sgn}(g_v)\right]} \\ \mu &= \frac{2\lambda mc}{\hbar} \sqrt{1 - \frac{E^2}{m^2 c^4}} \end{aligned} \tag{10}$$

and using (2)-(3) and (8)-(9) one obtains the differential Bessel equation

$$y^2 \psi'' + y \psi' + (y^2 - \mu^2) \psi = 0 \tag{11}$$

where the prime denotes differentiation with respect to  $y$ . The solution finite at  $y = 0$  ( $|x| = \infty$ ) is given by the Bessel function of the first kind and order  $\mu$  [6]:

$$\psi(y) = N_\mu J_\mu(y) \tag{12}$$

where  $N_\mu$  is a normalization constant. In fact, the normalizability of  $\psi$  demands that the integral  $\int_0^{y_0} y^{-1} |J_\mu(y)|^2 dy$  must be convergent. Since  $J_\mu(y)$  behaves as  $y^\mu$  at the lower limit, one can see that  $\mu \geq 1/2$  so that square-integrable KG eigenfunctions are allowed only if  $\lambda \geq \lambda_c/4$ , where  $\lambda_c = \hbar/(mc)$  is the Compton wavelength. The boundary conditions at  $x = 0$  ( $y = y_0$ ) imply that

$$\begin{aligned} \frac{dJ_\mu(y)}{dy} \bigg|_{y=y_0} &= 0, \quad \text{for even states} \\ J_\mu(y_0) &= 0, \quad \text{for odd states} \end{aligned} \tag{13}$$

Since the KG eigenenergies are dependent on  $\mu$  and  $y_0$ , it follows that Eq. (13) is a quantization condition. The allowed values for the parameters  $\mu$  and  $y_0$ , and  $E$  as an immediate consequence, are determined by solving Eq. (13). The oscillatory character of the Bessel function and the finite range for  $y$  ( $0 < y \leq y_0$ ) imply that there is a finite number of discrete KG eigenenergies. Of course, the number of bound-state solutions increases as  $y_0$  is increased. As we let  $\lambda \rightarrow \lambda_c/4$ , it can now be realized that the energy levels approach  $E = 0$  and tend to disappear one after another. It happens that an isolated zero-energy solution survives when  $\lambda = \lambda_c/4$ , regardless the relative values of the coupling constants (recall that  $g_s > 0$ ).

The roots of  $J_\mu(y)$  and  $J'_\mu(y)$  are listed in tables of Bessel functions only for a few special values of  $\mu$ . A bit of time and effort can be saved in the numerical calculation of the roots of  $J'_\mu(y)$  if one uses the recurrence relation  $J_{\mu-1} - J_{\mu+1} = 2J'_\mu$ , in such a manner that the quantization condition for even states translates into  $J_{\mu+1}(y_0) = J_{\mu-1}(y_0)$ .

When  $g_s = g_v$  ( $g_s = -g_v$ ) the single-well potential is deeper (shallower) for positive-energy levels than that one for negative-energy levels. Thus, the capacity to hold bound states depends on the sign of the eigenenergy and one might expect that the number of positive (negative) energy levels is greater than the number of negative (positive) energy levels. By the way, the positive (negative) energy solutions are not to be promptly identified with the solutions for particles (antiparticles). Rather, whether it is positive or negative, an eigenenergy can be unambiguously identified with a bounded solution for a particle (antiparticle) only by observing if the energy level emerges from the upper (lower) continuum.

The KG eigenenergies are plotted in Fig. 1 for the four lowest bound states as a function of  $g_s$  for  $g_v = g_s$  and  $\lambda = 2\lambda_c$ . The eigenenergies for  $g_v = -g_s$  can be obtained by changing  $E$  by  $-E$ , as mentioned before. Note that in the

case illustrated in Fig. 1 the eigenenergies correspond to bounded solutions for particles. There are no energy levels for antiparticles. Also, note that the energy level corresponding to the ground-state solution ( $\psi$  even) always makes its appearance and that the number of energy levels grows with  $g_s$ . The spectrum consists of a finite set of energy levels of alternate parities. The nonrelativistic limit is only viable for  $g_s = g_v$  whereas the case  $g_s = -g_v$  is an essentially relativistic problem. Furthermore, one has  $E \simeq mc^2$  as long as  $g_s \ll 1$ .

## 2.2 The effective Morse-like potential ( $g_s^2 \neq g_v^2$ )

For this class, the existence of bound-state solutions permits us to distinguish two subclasses: 1)  $V_1 < 0$  and  $V_2 > 0$ , corresponding to  $g_s + g_v E / (mc^2) > 0$  and  $g_s > |g_v|$ ; 2)  $V_1 < |V_2|$  and  $V_2 < 0$ , corresponding to  $g_s + g_v E / (mc^2) > -(g_v^2 - g_s^2) / (4mc^2)$  and  $|g_s| < |g_v|$ . Included into the first (second) subclass is the case of a pure scalar (vector) coupling. The first subclass, as well as the second one on the condition that  $g_v > |g_s|$ , contain the nonrelativistic theory as a limiting case. On the contrary, i.e.  $g_s < |g_v|$  and  $g_v < 0$ , the theory is essentially relativistic. Let us define

$$\begin{aligned} z &= z_0 \exp\left(-\frac{|x|}{\lambda}\right), & z_0 &= \frac{\sqrt{g_s^2 - g_v^2}}{\hbar c} \\ \rho &= \frac{\lambda m}{\hbar^2 z_0} \left(g_s + \frac{E}{mc^2} g_v\right), & \nu &= \frac{\lambda mc}{\hbar} \sqrt{1 - \frac{E^2}{m^2 c^4}} \end{aligned} \quad (14)$$

so that

$$z\psi'' + \psi' + \left(-\frac{z}{4} - \frac{\nu^2}{z} + \rho\right)\psi = 0 \quad (15)$$

Note that  $\psi$  is a function of complex variable,  $z$ , if  $g_s^2 < g_v^2$ . Following the steps of Refs. [4] and [5], we make the transformation  $\psi = z^{-1/2}\phi$  to obtain the Whittaker equation [6]:

$$\phi'' + \left(-\frac{1}{4} + \frac{\rho}{z} + \frac{1/4 - \nu^2}{z^2}\right)\phi = 0 \quad (16)$$

whose solution vanishing at the infinity is written as  $\phi = N z^{\nu+1/2} e^{-z/2} M(a, b, z)$ , where  $N$  is a normalization constant and  $M$  is the regular confluent hypergeometric function with

$$a = \nu + \frac{1}{2} - \rho, \quad b = 2\nu + 1 \quad (17)$$

Thus,

$$\psi = Nz^\nu e^{-z/2}M(a, b, z) \quad (18)$$

For this class of effective potential there is no restriction on the size of  $\lambda$  in order to make the existence of a bounded solution possible as there is for the previous class. The KG eigenfunction is normalizable for any  $\nu$  as easy inspection shows. Therefore, one can think of a very short-ranged potential in the sense of  $\lambda \rightarrow 0$ , i.e. a potential approaching the  $\delta$ -function. Indeed, this sort of limit has already been realized by Domínguez-Adame and Rodríguez [1] for the case of a pure vector potential. From Eq. (18) one can see now that the boundary conditions at  $x = 0$  ( $z = z_0$ ) imply into the quantization conditions

$$\frac{M(a+1, b+1, z_0)}{M(a, b, z_0)} = \frac{z_0 - 2\nu}{2^{\frac{a}{b}} z_0}, \quad \text{for even states} \quad (19)$$

$$M(a, b, z_0) = 0, \quad \text{for odd states}$$

If  $g_s$  happens to vanish, the spectrum will only consist of positive (negative) energy levels for  $g_v > 0$  ( $g_v < 0$ ). If  $g_s \neq 0$ , though, the spectrum may acquiesce both signs of eigenenergies. The presence of both signs of eigenenergies depends, of course, on the relative strength between the vector and scalar potentials. When  $g_v = 0$  the negative- and positive-energy levels are disposed symmetrically about  $E = 0$ , as commented before, so that there are as many positive-energy levels as negative ones. In the case  $g_s > |g_v|$  it is reasonable to expect a two-fold degeneracy as  $g_s/|g_v| \rightarrow \infty$  due to the double-well structure with an infinitely high barrier potential between the wells. That degeneracy in an one-dimensional quantum-mechanical problem is due to the fact that even eigenfunctions tend to vanish at the origin as  $g_s/|g_v| \rightarrow \infty$ .

The Fig. 2 illustrates the four lowest states of the spectrum for this class of effective potential as a function of  $g_s/|g_v|$  with  $g_v > 0$ . The energy level corresponding to the ground-state solution ( $\psi$  even) always makes its appearance. As before, the eigenenergies for  $g_v < 0$  can be obtained by replacing  $E$  by  $-E$  (recall that  $g_s > -|g_v|$ ). Note that only bounded solutions for particles are present for  $g_s/|g_v| < 1$  (even if  $E < 0$ ) and that a new branch of solutions corresponding to antiparticles emerges from the lower continuum as  $g_s/|g_v|$  increases starting from  $g_s/|g_v| = 1$ . In that last case as  $g_s/|g_v| \rightarrow \infty$  the even

and odd parities solutions tend to be degenerate and the spectrum tends to exhibit a symmetry about  $E = 0$ .

### 3 Conclusions

Using the same method used in prior works, we have succeed in the proposal of searching the solution for a more general screened Coulomb potential with the KG equation. An opportunity was given by that generalization to analyze some aspects of the KG equation which would not be feasibly only with the special cases already approached in the literature. Thus, the use of the mixing of vector and scalar Lorentz structures for other kinds of potentials may lead to a better understanding of the KG equation and its solutions. Free from doubt, this sort of mixing also deserves to be explored with the Dirac equation.

It is worthwhile to mention that the solutions of the KG equation with a screened Coulomb potential present a continuous transition as the ratio  $g_s/g_v$  varies. However, a phase transition occurs when  $|g_s/g_v| = 1$ . Although the phase transition does not always show its face for the KG eigenenergies (observe carefully the continuity of the KG eigenenergies for the particle energy levels in Fig. 2), it clearly shows it for the KG eigenfunctions (note, for instance, that the behaviour of the KG eigenfunction for both classes of effective potentials differ at the neighborhood of the origin).

Finally, we draw attention to the fact that no matter how strong the potentials are, as far as  $g_s \geq -|g_v|$ , the energy levels for particles (antiparticles) never dive into the lower (upper) continuum. Thus there is no room for the production of particle-antiparticle pairs. This all means that Klein's paradox never comes to the scenario.

### Acknowledgments

This work was supported in part by means of funds provided by CNPq and FAPESP.

## References

- [1] F. Domínguez-Adame, A. Rodríguez, Phys. Lett. A 198 (1995) 275.
- [2] B.A. Kagali, N.A. Rao, V. Sivramkrishna, Mod. Phys. Lett. A 17 (2002) 2049.
- [3] A.S. de Castro, Ann. Phys. (N.Y.), in press.
- [4] N.A. Rao, B.A. Kagali, Phys. Lett. A 296 (2002) 192.
- [5] N.A. Rao, B.A. Kagali, V. Sivramkrishna, Int. J. Mod. Phys. A 17 (2002) 4793.
- [6] M. Abramowitz, I.A. Stegun, Handbook of Mathematical Functions, Dover, Toronto, 1965.

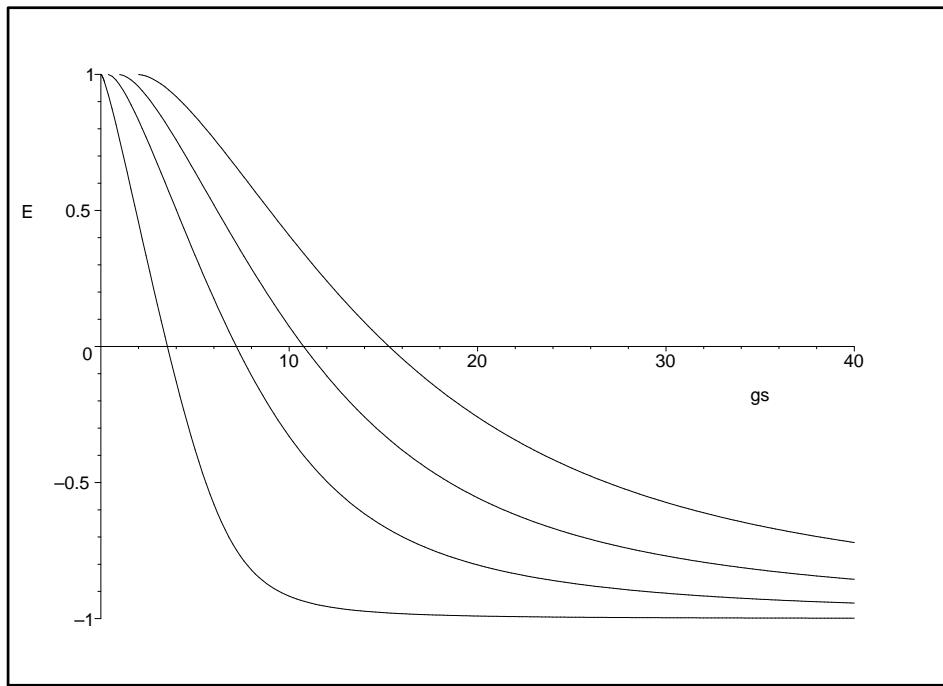


Figure 1: KG eigenenergies for the four lowest energy levels as a function of  $g_s$  for  $g_v = g_s$  ( $\lambda = 2\lambda_c$  and  $m = c = \hbar = 1$ ).

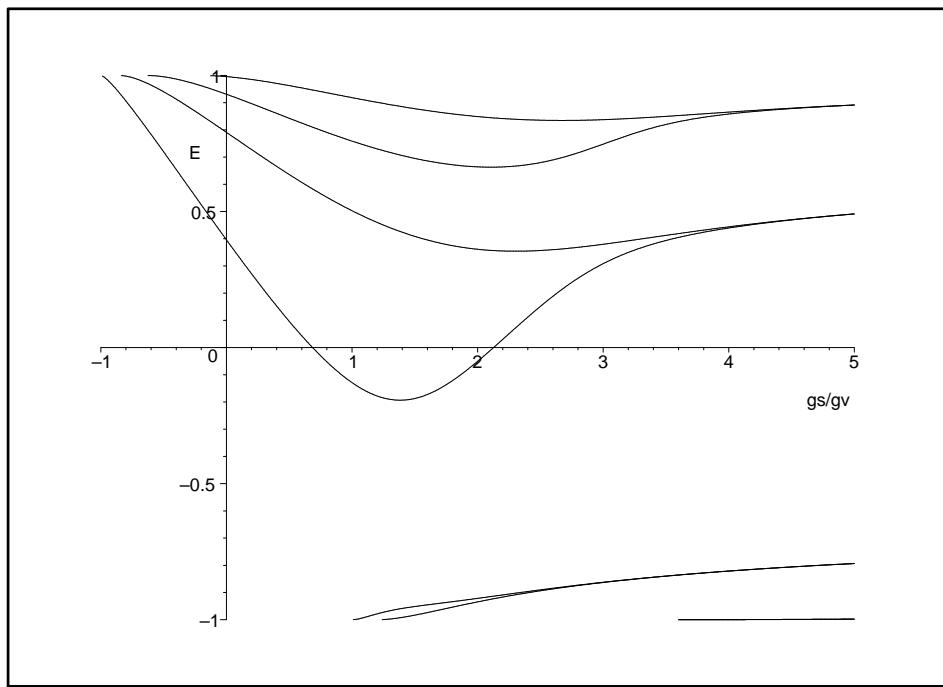


Figure 2: KG eigenenergies for the four lowest energy levels as a function of  $g_s/g_v$  ( $g_v = 4*$ ,  $\lambda = 2\lambda_c$  and  $m = c = \hbar = 1$ ).